Exponential decay of relative entropies to the Kolmogorov-Sinai entropy for the standard map

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It is shown how to obtain a fast convergence of the relative dynamical entropies to the limiting Kolmogorov-Sinai entropy for the generating partition of the standard map.

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In a recent interesting paper Christiansen and Politi found a presumably generating partition for the standard map [1]. This partition, consisting of seven disjoint parts of the phase space, is constructed from the primary homoclinic tangencies. A generating partition $\mathcal{P}$ for a map T consisting of N parts allows one to establish an N letters symbolic dynamics and to compute the Kolmogorov-Sinai entropy H_{KS} of T as [2,3]

$$H_{\mathrm{KS}}=\lim_{k\to\infty}\frac{1}{k}H_k \ . \tag{1}$$

The partial entropy H_k/k is given by

$$\begin{aligned}H_k:&=H\left[\bigvee_{j=0}^{k-1}T^{-j}\mathcal{P}\right]\\&=-\sum_{i_0,\ldots,i_{k-1}}p(i_0,\ldots,i_{k-1})\ln p(i_0,\ldots,i_{k-1})\ ,\end{aligned} \tag{2}$$

where $p(i_0,\ldots,i_{k-1})$ is the probability of occurrence of the symbols sequence $i_0,\ldots,i_{k-1}$, and the sum is taken over all possible N^k sequences. Here, $H(\mathcal{Q})$ denotes the entropy of a partition $\mathcal{Q}$, and the elements of the partition $\bigvee_{j=0}^{k-1}T^{-j}\mathcal{P}$ are all sets of the form $\bigcap_{j=0}^{k-1}T^{-j}A_j$, where $A_j\in\mathcal{P}$ for $j=0,\ldots,k-1$.

The convergence to the limit (1) is usually slow, not faster than $1/k$. It is therefore advantageous [4–6] to consider the relative entropies G_k defined as

$$G_k:=H\left[\mathcal{P}\,\middle|\,\bigvee_{j=0}^{k-1}T^{-j}\mathcal{P}\right]=H_k-H_{k-1}\quad\text{for } k>1;$$
$$G_1:=H_1\ , \tag{3}$$

where $H(\mathcal{Q}|\mathcal{S})$ denotes the relative entropy of a partition $\mathcal{Q}$ with respect to a partition $\mathcal{S}$.

It is easy to see that the sequence G_k also tends to H_{KS} [2,3]. It seems that this limit is achieved much faster than the limit (1). In fact, Misiurewicz and Ziemian [5] showed that for a certain class of maps from the unit interval onto itself this convergence is exponential. Such a convergence was also reported by Szépfalusy and Györgyi [4] for the bilinear, the biquadratic, and the piecewise linear maps on the interval. Note that the convergence of $(1/k)H_k=1/k\sum_{n=1}^{k}G_n$ is slower, since the terms of larger n have to balance a poor precision of the approximation obtained out of the first terms.

To verify the properties of the partition constructed for the standard map, Christiansen and Politi [1] calculated the partial entropies H_k/k and compared them with the value of the KS entropy estimated via Pesin formula $H_{\mathrm{KS}}\approx 1.1365$. Using their results for H_k we compute the relative entropies G_k and show the results in Table I. The sequence of the relative entropies G_k tends rapidly to the limiting value of the KS entropy. In order to get a better approximation (e.g., G_{10}) one requires the partial entropies H_k computed with the precision higher than in Ref. [1]. However, one can try to evaluate the KS entropy assuming that the relative entropies G_k decay exponentially. A numerical fit $G_k\sim G_\infty+a\gamma^k$ to the data of Table I gives the extrapolation $G_\infty\approx 1.13$ with the decay rate $\gamma\approx 0.66$.

Out of the data presented in Ref. [1] we have extracted a better approximation for the dynamical entropy of the standard map. This helps one to appreciate the significance of the work of Christiansen and Politi.

TABLE I. Partial entropies H_k/k taken from Ref. [1] and relative entropies G_k defined in Eq. (3) for lengths $k\le 9$.

k	H_k/k	G_k
1	1.77292	1.773
2	1.69554	1.618
3	1.59844	1.404
4	1.52601	1.308
5	1.47195	1.255
6	1.43028	1.221
7	1.39680	1.195
8	1.36969	1.179
9	1.3475	1.169

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